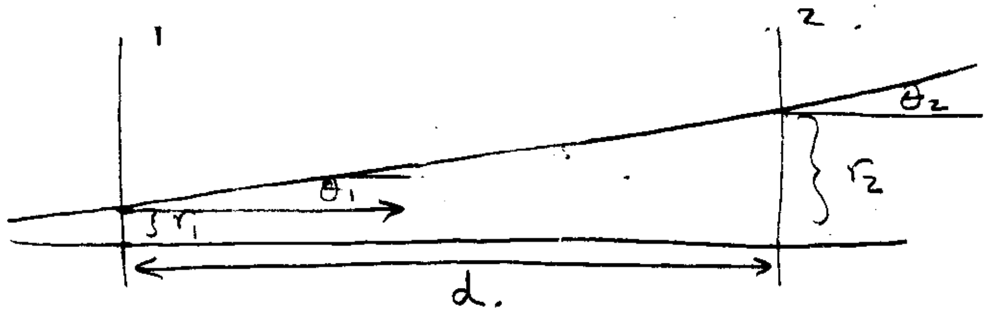
If the angle θ is small the beam hardly changes size. $\Rightarrow$ we can treat the beam as a ray.

Ray - Path that the center of a slowly diverging EM beam takes as it goes through a system.
- does not have an amplitude
- spatial extent of beam is small compared to the size of the components.

Notice that $\theta \simeq \frac{\lambda}{\pi w_0}$ if $\lambda \ll$ size of optics
$\Rightarrow$ θ is small, beam is slowly diverging wrt optics.

Ray Tracing.



Homogeneous dielectric of length d. (free space).

Assume ray is paraxial - at small angle wrt axis
$\tan\theta \approx \sin\theta \sim \theta$

Nothing changes beam so:

$$r_2 = 1 \cdot r_1 + d\, r_1'$$
$$r_2' = 0 \cdot r_1 + 1 \cdot r_1' \qquad (r_2' = r_1').$$

r_2' & r_1' are the slope of the ray.

We can write this in matrix form:

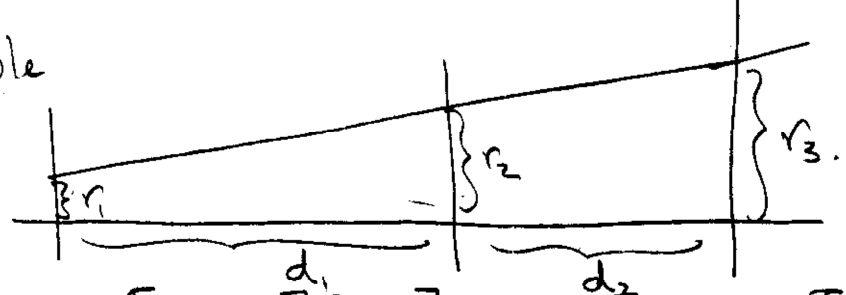
$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

In general we write the relationship between input and output an ABCD matrix:

$$\begin{bmatrix} r_{out} \\ r_{out}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r_{in}' \end{bmatrix}.$$

2 X 2 matrix. describes system.

For example



$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ r_2' \end{bmatrix}, \quad \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}.$$

$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1 & d_1+d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

as you should expect.

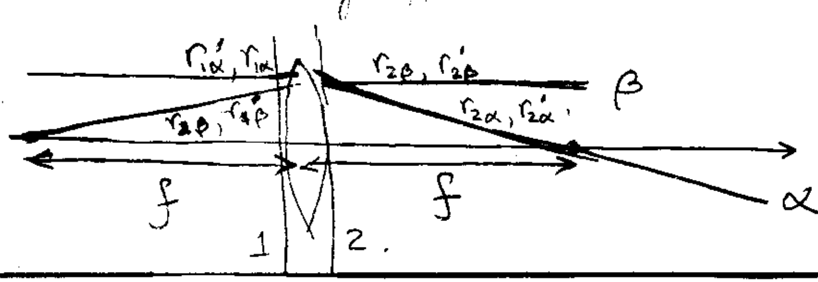
$d \rightarrow d_1 + d_2$.

Common Ray Matrices.

Thin lens of focal length f.

assume lens is so thin that distance between entrance and exit is negligible.

$r_2 = r_1 \Rightarrow A = 1, B = 0$



$r_{2\alpha} = r_{1\alpha}$ $\quad r'_{2\alpha} = C r_{1\alpha} + D r'_{1\alpha} = -\frac{r_{1\alpha}}{f}$

$$C = -\frac{1}{f} \quad D = 0.$$

$r_{2\beta} = r_{1\beta}$ $\quad r'_{2\beta} = 0 = -\frac{1}{f} r_{1\beta} + D r'_{1\beta}$ $\quad r'_{1\beta} = \frac{r_{1\beta}}{f}.$

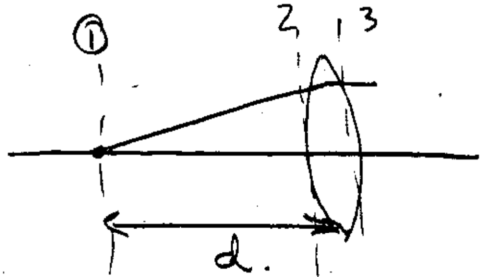
$$D = 1.$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}.$$

true for all thin lens
$f > 0$ converging
$f < 0$ diverging.

What about.



free space + thin lens.

$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} \qquad \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}.$$

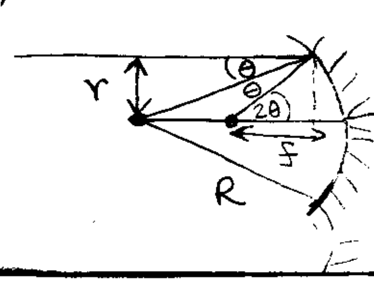
$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix},$$

$$= \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}.$$

* Notice order is important
last element is first matrix.
non-commuting.

<u>Spherical Mirror</u>.

(1) ray ∥ to mirror axis is reflected to focus
(2) ∡ incidence = ∡ reflection.



$\sin\theta = \frac{r}{R}$

$\tan 2\theta \simeq \sin 2\theta = \frac{r}{f}$

For paraxial $\theta \simeq \sin\theta$

$$\Longrightarrow \quad \theta \simeq \frac{r}{R} \qquad 2\theta \simeq \frac{r}{f} \qquad R = 2f \qquad f = \frac{R}{2}.$$

Mirror.

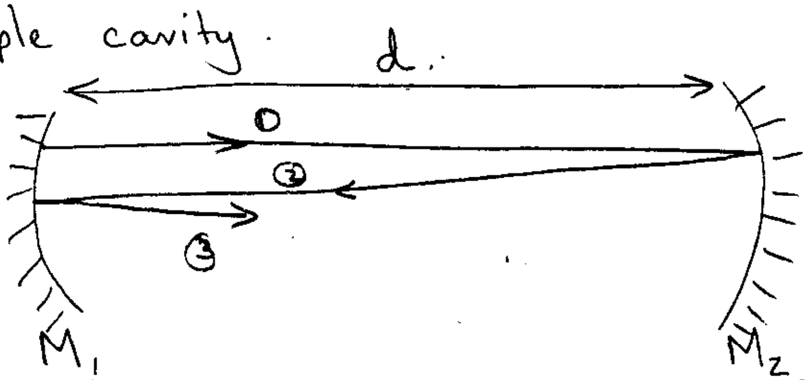
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}.$$

Notice $R = \infty$
$\Rightarrow$ flat mirror.
$r_{out} = r_{in}$
$r'_{out} = r'_{in}$

Slope of ray is relative to direction

Optical Cavities.

Simple cavity.



We want ray to bounce back and forth between mirrors and not get out.

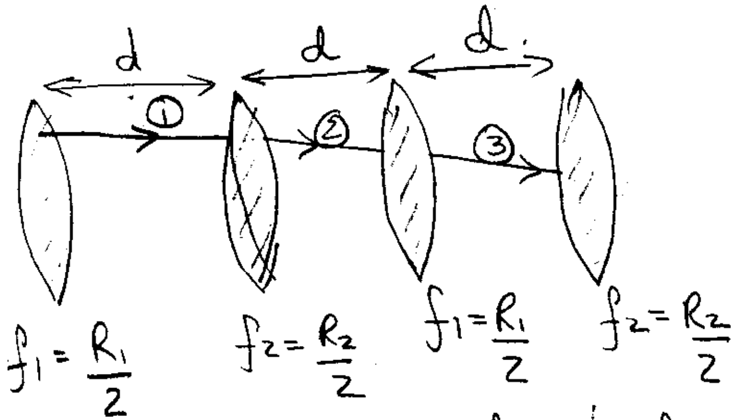
Notice that for "thin lens" $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$.

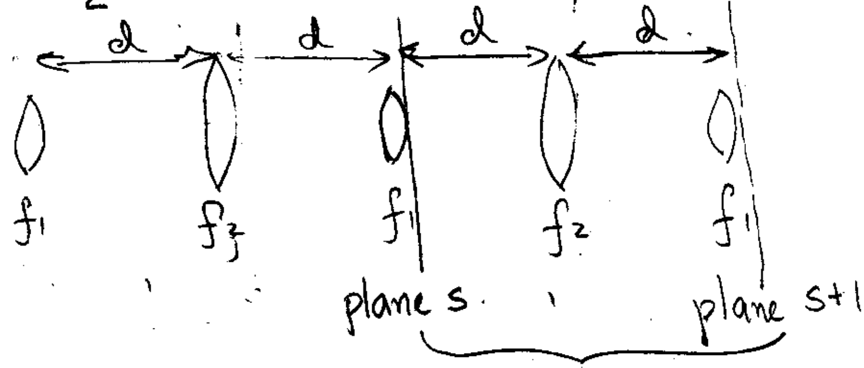
spherical mirror $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$.

Spherical mirror is like a lens of focal length

$$f = \frac{R}{2}.$$

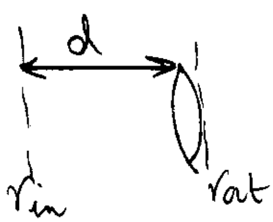
Therefore our ray above works like it is in a lens system.





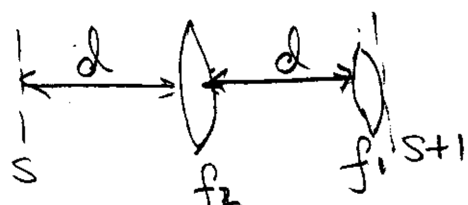
unit cell - one round trip through the resonator.

Recall that for



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1-\frac{d}{f} \end{bmatrix}.$$

$\therefore$ for unit cell



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ -\frac{1}{f_1} & 1-\frac{d}{f_1} \end{bmatrix} \begin{bmatrix} 1 & d \\ -\frac{1}{f_2} & 1-\frac{d}{f_2} \end{bmatrix}.$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - d/f_2 & d + d\left(1 - d/f_2\right) \\ -\frac{1}{f_1} - \frac{1}{f_2}\left(1-\frac{d}{f_1}\right) & \left(1-\frac{d}{f_1}\right)\left(1-\frac{d}{f_2}\right) - \frac{d}{f_1} \end{bmatrix}$$

$A = 1 - \frac{d}{f_2}$ $\qquad$ $C = -\left[\frac{1}{f_1} + \frac{1}{f_2}\left(1-\frac{d}{f_1}\right)\right]$

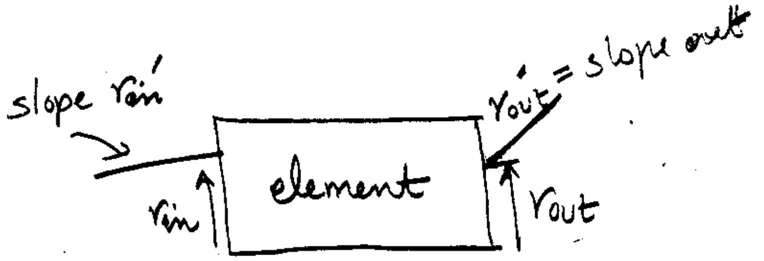
$B = d\left(2 - \frac{d}{f_2}\right)$ $\qquad$ $D = -\frac{d}{f_1} + \left(1-\frac{d}{f_1}\right)\left(1-\frac{d}{f_2}\right)$

Review.

Optical Ray Tracing.

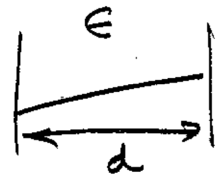
- paraxial approximation.

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}.$$



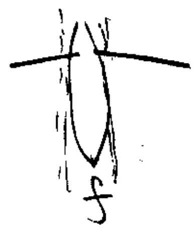
We derived a few ABCD matrices.

Dielectric medium



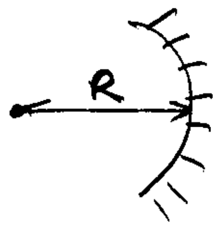
$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

lens - thin



$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

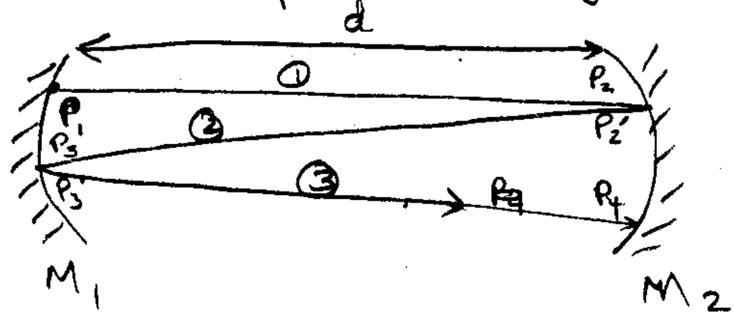
Spherical Mirror.



$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}.$$

looks like a thin lens w $f = \frac{R}{2}$.

Use these results in an optical cavity:



$P_1 \to P_2$ $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

$P_2 \to P_2'$ $\begin{bmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{bmatrix}$

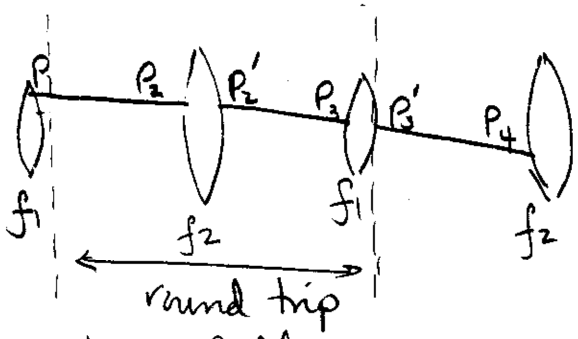
$P_2' \to P_3$ $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$.

$P_3 \to P_3'$ $\begin{bmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{bmatrix}$.

one round trip

$P_3' \to P_4$ $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$.

Spherical Mirrors are like lens of $f = \frac{R}{2}$.



For this unit Cell.

T - transmission matrix is given by the product

$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$P_3' \longleftarrow P_3 \longleftarrow P_2' \longleftarrow P_2 \leftarrow P_1$$

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ where $A = 1 - \frac{d}{f_2}$ $\quad B = d\left(2 - \frac{d}{f_2}\right)$

$$C = -\left[\frac{1}{f_1} + \frac{1}{f_2}\left(1 - \frac{d}{f_1}\right)\right]$$

$$D = -\frac{d}{f_1} + \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right).$$

Note that for N round trips you would have

T^N to calculate total transmission
~~or~~ positn of ray after N round trips.

We only care to know the position of the ray on say mirror M_1, at the end of each round trip.

But notice, rather than calculating T^N we can see that

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{T\text{-transmission matrix.}} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}.$$

$$r_{s+1} = A r_s + B r'_s \quad ①$$
$$r'_{s+1} = C r_s + D r'_s \quad ②$$
} plane $s \rightarrow s+1$.

Is it possible to relate r_{s+2} to r_{s+1} and r_s (diff eqn).

From ① $\quad r'_s = \frac{1}{B}\left(r_{s+1} - A r_s\right)$. ③.

Let $s \rightarrow s+1$. $\quad r'_{s+1} = \frac{1}{B}\left(r_{s+2} - A r_{s+1}\right)$ ④.

use ③ and ④ in ②.

$$\frac{1}{B}\left(r_{s+2} - A r_{s+1}\right) = C r_s + \frac{D}{B}\left(r_{s+1} - A r_s\right).$$

$$r_{s+2} - (A+D) r_{s+1} + \underbrace{(AD-BC)}_{=1} r_s = 0.$$

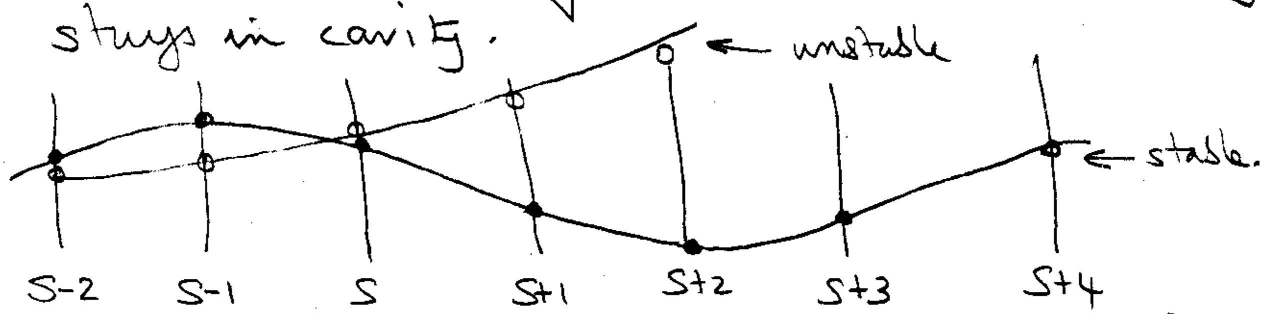
for any $\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}$ matrix with same index of refracti at entrance and exit planes.
∴ always true for optical cavities.

let $b = \frac{1}{2}(A+D)$.

$$\boxed{r_{s+2} - 2b\, r_{s+1} + r_s = 0}.$$

difference equation

Remember we are interested in beams that stay in the cavity, so we want solutions for r_s that are oscillating → ray stays in cavity.



Or at least, we want to know how long the ray stays in the cavity, how many round trips.

If rays position is not bounded, it will eventually miss the mirror and exit the bi periodic lens system (cavity).

Assume solutions must be of form (stay in cavity).

$$r_s = r_{max} \sin(s\theta + \alpha) \quad ; \quad \alpha = \text{initial condit.}$$
$$= r_{max}\, \mathrm{Im}\left(e^{j(s\theta+\alpha)}\right).$$

Substitute this into difference equation.

$$r_{max}\, e^{j((s+2)\theta+\alpha)} - 2b r_{max}\, e^{j(s+1)\theta+\alpha} + r_{max}\, e^{j(s\theta+\alpha)} = 0.$$

True if

$$e^{j2\theta} - 2be^{j\theta} + 1 = 0.$$

$$(e^{j\theta})^2 - 2be^{j\theta} + 1 = 0.$$

quadratic equation $x = e^{j\theta}$ $\quad x^2 - 2bx + 1 = 0.$

$$e^{j\theta} = x = \frac{2b \pm \sqrt{4b^2-4}}{2} = b \pm j\sqrt{1-b^2}.$$

$$e^{j\theta} = \cos\theta + j\sin\theta. \Rightarrow b = \cos\theta = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2}$$

or $$\theta = \cos^{-1}\left[1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2}\right].$$

Notice: for r_s to be oscillating then θ must be real.

if θ is imaginary, ray does not oscillate (say $\theta = j\phi$)

$$r_s = r_{max}\, \mathrm{Im}\left(e^{s\phi} e^{j\alpha}\right) = r_{max}\, e^{s\phi} \sin\alpha.$$

as $s\uparrow$, $r_s\uparrow$ and as $s\to\infty$, $r_s\to\infty$.

$\therefore$ for stability, i.e. for r_s to be bounded by r_{max} we require θ to be real.

For stability it is sufficient to require that

$$-1 \le \cos\theta \le 1$$

$$-1 \le b \le 1 \qquad b = \tfrac{1}{2}(A+D).$$

add 1 & divide by 2.

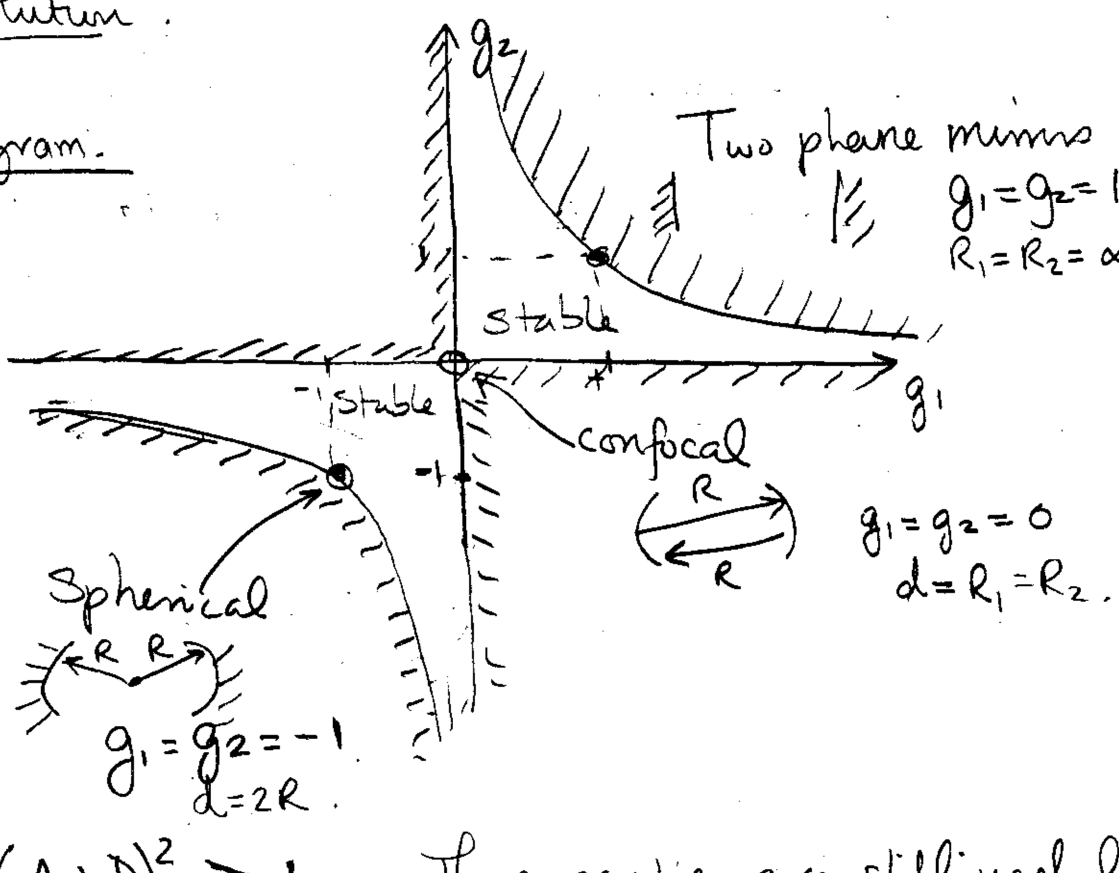
$$0 \le 1 - \frac{d}{2f_2} - \frac{d}{2f_1} + \frac{d^2}{f_1 f_2} \le 1$$

$$0 \le g_1 g_2 \le 1$$

$$g_1 = 1 - \frac{d}{2f_1} = 1 - \frac{d}{R_1} \qquad g_2 = 1 - \frac{d}{2f_2} = 1 - \frac{d}{R_2}.$$

Back to our solution.

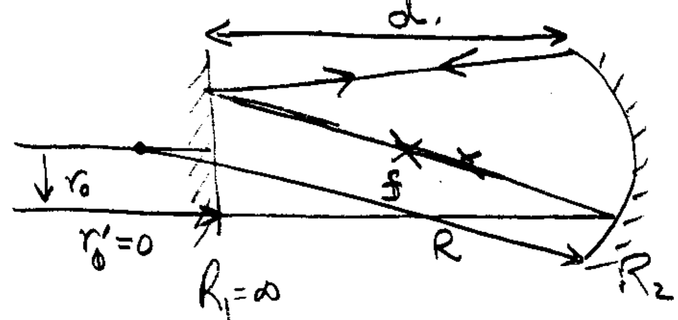
Stability Diagram.



Unstable. $\left(\frac{A+D}{2}\right)^2 > 1$ These cavities are still useful (unstable but we can use them).

Example of ray-tracing in a stable cavity.

- example in text - try to show details



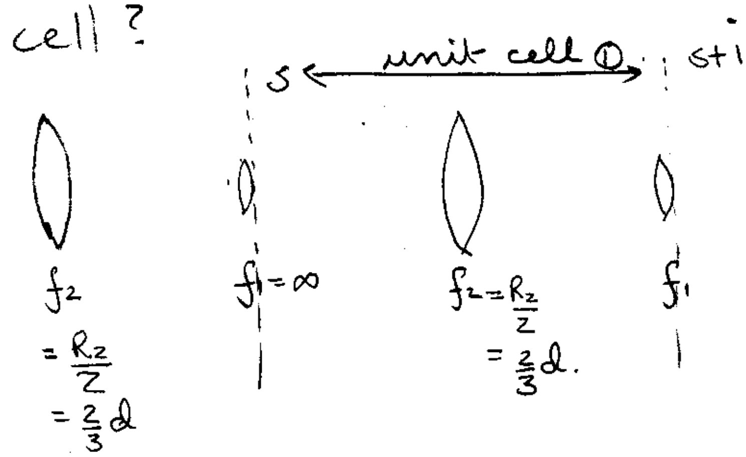
$$\frac{d}{R_2} = \frac{3}{4}$$

Stability .. $R_1 = \infty \quad R_2 = \frac{4}{3} d.$

$$g_1 = 1 - \frac{d}{R_1} = 1 \qquad g_2 = 1 - \frac{d}{R_2} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$g_1 g_2 = \frac{1}{4} < 1 \implies \text{Stable cavity.}$$

What is unit cell?



What does $f_1 = \infty$ mean?
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{identity}}$$

output = input.
does nothing.

$$\therefore\ T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ -\frac{1}{f_2} & -\frac{d}{f_2}+1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f_2} & d + d\left(1 - \frac{d}{f_2}\right) \\ -\frac{1}{f_2} & 1 - d/f_2. \end{bmatrix}$$