$S_q \propto$ electric fields at other frequencies.

$\Rightarrow$ waves are coupled. If $d=0$, nonlinearity vanishes.
$\Rightarrow$ normal Helmholtz equation.

Notice, if $\omega_1, \omega_2, \omega_3$ are not commensurate (one frequency is not the sum or difference of the other two), the S has no sources at ω_1, ω_2, and ω_3.
$\Rightarrow$ S vanishes and waves do not interact.

For the three waves to be coupled by the medium, frequencies must commensurate:

$$\boxed{\omega_3 = \omega_1 + \omega_2} \quad \text{frequency matching condition.}$$

or $\omega_3 = \omega_1 - \omega_2$.

Example, if $\omega_3 = \omega_1 + \omega_2$.

$$\left.\begin{aligned} S_1 &= 2\mu_0 \omega_1^2 d E_3 E_2^* \\ S_2 &= 2\mu_0 \omega_2^2 d E_3 E_1^* \\ S_3 &= 2\mu_0 \omega_3^2 d E_1 E_2 \end{aligned}\right\} \text{waves are coupled.}$$

($E_3 E_2^*$ has frequency $\omega_1 = \omega_3 - \omega_2$ etc).

$$\boxed{\begin{aligned} (\nabla^2 + k_1^2) E_1 &= -2\mu_0 \omega_1^2 d E_3 E_2^* \\ (\nabla^2 + k_2^2) E_2 &= -2\mu_0 \omega_2^2 d E_3 E_1^* \\ (\nabla^2 + k_3^2) E_3 &= -2\mu_0 \omega_3^2 d E_1 E_2 \end{aligned}} \quad \text{Three-Wave Mixing Coupled Equations}$$

Last time we showed that the coupled three-wave mixing equations are:

$$(\nabla^2 + k_1^2)E_1 = -2\mu_0\omega_1^2 d E_3 E_2^*$$
$$(\nabla^2 + k_2^2)E_2 = -2\mu_0\omega_2^2 d E_3 E_1^*$$
$$(\nabla^2 + k_3^2)E_3 = -2\mu_0\omega_3^2 d E_1 E_2$$

for $\omega_3 = \omega_1 + \omega_2$ (frequency matching condition).

Assume:

$$\begin{array}{l} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \quad E_q = A_q \exp(-jk_q z) \qquad A_q \text{ is complex}$$
$$k_q = \frac{\omega_q}{c} \qquad q = 1, 2, 3.$$

Normalize amplitude: $a_q = \dfrac{A_q}{(2\eta\hbar\omega_q)^{1/2}}$ $\quad \eta = \dfrac{\eta_0}{n}$ $\quad \eta_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}}$

$\hbar\omega_q$ - energy of photon of angular frequency ω_q.

$$\Rightarrow \quad E_q = (2\eta\hbar\omega_q)^{1/2} a_q \exp(-jk_q z) \qquad q = 1, 2, 3.$$

Intensity $\quad I_q = \dfrac{|E_q|^2}{2\eta} = \hbar\omega_q |a_q|^2$

Photon flux density (photons/s-m²).

$$\phi_q = \frac{I_q}{\hbar\omega_q} = |a_q|^2$$

we use this notation because the wave mixing depends on the photon-number conservation.

Because of interaction, a_q will vary with z i.e. $a_q = a_q(z)$.

Assume $a_q(z)$ varies slowly with z. (approximately constant within a distance of a wavelength.

(Slowly varying envelope approximation)

$\Rightarrow$ $\frac{d^2 a_q}{dz^2}$ is neglected relative to $k_q \frac{da_q}{dz} = \left(\frac{2\pi}{\lambda_q}\right) \frac{da_q}{dz}$.

$$\Rightarrow (\nabla^2 + k_q^2)\left[a_q \exp(-jk_q z)\right] \simeq -j2k_q \frac{da_q}{dz} \exp(-jk_q z).$$

(verify this for yourself).

$$\Rightarrow \left.\begin{aligned} \frac{da_1}{dz} &= -jg\, a_3 a_2^* \exp(-j\Delta k z) \\ \frac{da_2}{dz} &= -jg\, a_3 a_1^* \exp(-j\Delta k z). \\ \frac{da_3}{dz} &= -jg\, a_1 a_2 \exp(j\Delta k z). \end{aligned}\right\} \quad \begin{aligned} g^2 &= 2\hbar\omega_1\omega_2\omega_3 \eta^3 d^2 \\ \Delta k &= k_3 - k_2 - k_1. \end{aligned}$$

a_1, a_2, a_3 - governed by three coupled first-order differential equations.

We can further show that $\frac{d}{dz}(I_1 + I_2 + I_3) = 0$. (Energy Conservation)

and Photon-number conservation: Manley-Rowe Relation:

$$\frac{d}{dz}|a_1|^2 = \frac{d}{dz}|a_2|^2 = -\frac{d}{dz}|a_3|^2.$$

or $$\frac{d\phi_1}{dz} = \frac{d\phi_2}{dz} = -\frac{d\phi_3}{dz}$$

or $$\frac{d}{dz}\left(\frac{I_1}{\omega_1}\right) = \frac{d}{dz}\left(\frac{I_2}{\omega_2}\right) = -\frac{d}{dz}\left(\frac{I_3}{\omega_3}\right).$$ ⇐ Manley-Rowe relation.

This implies that $|a_1|^2+|a_3|^2$ and $|a_2|^2+|a_3|^2$ are invariant in wave-mixing processes.

Second Harmonic Generation.

$$\omega_1=\omega_2=\omega \quad \text{and} \quad \omega_3=2\omega.$$

Two interactions can occur:

① Two photons of freq ω combine to form a photon of frequency 2ω (second harmonic).

② One photon of frequency 2ω splits into 2 photons of frequency ω.

Conservation of momentum ⇒ $\vec{k}_3 = 2\vec{k}_1$ (phase matching)

Two waves of amplitude E_1 & E_3.

$$\Rightarrow \quad S_1 = 2\mu_0\,\omega_1^2\, d\, E_3 E_1^*$$
$$S_2 = \mu_0\,\omega_3^2\, d\, E_1 E_1.$$

$$(\nabla^2+k_1^2)\,E_1 = -2\mu_0\,\omega_1^2\, d\, E_3 E_1^*$$
$$(\nabla^2+k_3^2)\,E_3 = -\mu_0\,\omega_3^2\, d\, E_1 E_1.$$

$$\therefore \quad \frac{da_1}{dz} = -j\, g\, a_3 a_1^* \exp(-j \Delta k z).$$

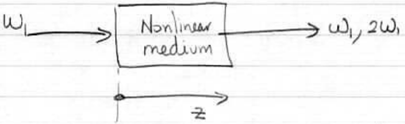
$$\frac{da_3}{dz} = -j \frac{g}{2} a_1 a_1 \exp(-j \Delta k z) \qquad \Delta k = k_3 - 2k_1$$

$$g^2 = 4\hbar \omega^3 n^3 d^2$$

Assuming two collinear waves with perfect phase matching

$(\Delta k = 0) \quad \Rightarrow \quad \dfrac{da_1}{dz} = -j g a_3 a_1^*$

$$\frac{da_3}{dz} = -j \frac{g}{2} a_1 a_1.$$



$$\underbrace{a_3(0) = 0. \qquad a_1(0) \text{ assumed to be real.}}_{\text{Boundary conditions.}}$$

and $|a_1(z)|^2 + 2|a_3(z)|^2 = \text{constant}$

$$\Rightarrow \quad a_1(z) = a_1(0) \operatorname{sech}\left(\frac{g a_1(0) z}{\sqrt{2}}\right).$$

$$a_3(z) = \frac{-j}{\sqrt{2}} a_1(0) \tanh\left(\frac{g (a_1(0)) z}{\sqrt{2}}\right).$$

$$\Rightarrow \quad \phi_1(z) = \phi_1(0) \operatorname{sech}^2\left(\frac{\gamma z}{2}\right)$$

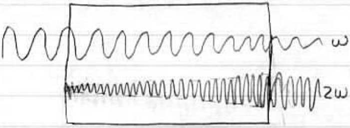
$$\phi_2(z) = \frac{1}{2} \phi_1(0) \tanh^2\left(\frac{\gamma z}{2}\right). \; ; \quad \frac{\gamma}{2} = \frac{g a_1(0)}{\sqrt{2}}$$

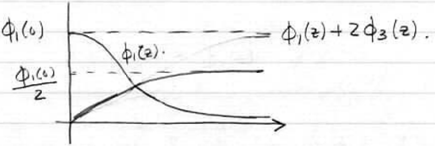
$$\gamma^2 = 2g^2 a_1^2(0) = 2g^2\phi_1(0) = 8d^2\eta^3\hbar\omega^3\phi_1(0) = 8d^2\eta^3\omega^3 I_1(0).$$

Notice $\quad \mathrm{sech}^2 + \tanh^2 = 1.$

$$\Rightarrow \quad \phi_1(z) + 2\phi_3(z) = \phi_1(0) = \text{constant}.$$

photons of wave 1 are converted to half as many photons of wave 3.





Second harmonic efficiency :

$$\frac{I_3(L)}{I_1(0)} = \frac{\hbar\omega_3\phi_3(L)}{\hbar\omega_1\phi_1(0)} = \frac{2\phi_3(L)}{\phi_1(0)}$$

$$\boxed{\frac{I_3(L)}{I_1(0)} = \tan^2\frac{\gamma L}{2}.}$$

For large $\gamma L = \sqrt{8}\, d\, \eta^{3/2}\omega^{3/2} I_1^{1/2}(0)\, L$ efficiency approaches 1.

$\Rightarrow$ all power converted to second harmonic.

For small γL (small device length L, small nonlinear parameter d, or small input $\phi_1(0)$, $\tanh(x) \simeq x$ for small x.

$$\Rightarrow \tanh^2\left(\frac{\gamma L}{2}\right) \simeq \frac{\gamma^2 L^2}{4}$$

- $$\frac{I_3(L)}{I_1(0)} = \frac{1}{4}\gamma^2 L^2 = \frac{1}{2} g^2 L^2 \phi_1(0) = 2d^2\eta^3\hbar\omega^3 L^2 \phi_1(0) = 2d^2\eta^3\omega^2 L^2 I_1(0)$$

$$\frac{I_3(L)}{I_1(0)} = 2\eta_0^3\omega^2 \frac{d^2}{\eta^3}\frac{L^2}{A} P.$$

$P = I_1(0)A$ is the incident optical power and A is the cross-sectional area.

$\frac{d^2}{\eta^3}$ – figure of merit. (material dependent)

$\frac{L^2}{A}$ geometrical factor.

Effect of phase mismatch $\quad \Delta k \neq 0$.

- Weak coupling case $\gamma L \ll 1$. $a_1(z)$ varies slightly with z. and $a_1(z) \simeq a_1(0)$ (approximately constant).

$$\Rightarrow \frac{da_3}{dz} = -j\frac{g}{2} a_1^2(0) \exp(j\Delta k z)$$

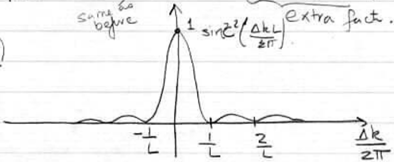
Integrate over z from 0 to L

$$\Rightarrow a_3(L) = -\left(\frac{g}{2\Delta k}\right) a_1^2(0)\left[\exp(j\Delta k L) - 1\right]$$

$$\phi_3(L) = |a_3(L)|^2 = \left(\frac{g}{\Delta k}\right)^2 \phi_1^2(0)\sin^2\left(\frac{\Delta k L}{2}\right).$$

or $$\frac{I_3(L)}{I_1(0)} = \frac{2\phi_3(L)}{\phi_1(0)} = \underbrace{\frac{1}{2}g^2L^2\phi_1(0)}_{\text{same as before}}\ \underbrace{\operatorname{sinc}^2\left(\frac{\Delta k L}{2\pi}\right)}_{\text{extra fact.}}.$$

- $$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$



If $|\Delta k| = \frac{\pi}{L} \Rightarrow 0.4 =$ factor. phase matching is more stringent as L becomes longer.

For a given Δk $\quad L_c = \frac{2\pi}{|\Delta k|}$ - measure of maximum length within which 2nd Harmonic generation is efficient.

L_c - coherence length.

$$|\Delta k| = 2\left(\frac{2\pi}{\lambda_0}\right)|n_3-n_1|$$

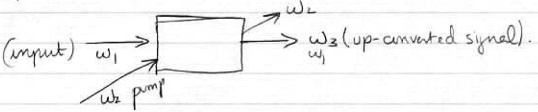
λ_0 - free space wavelength of fundamental wave

$$L_c = \frac{\lambda_0}{2}|n_3-n_1|$$

inversely proportional to $\underbrace{|n_3-n_1|}_{\text{material dispersion}}$.

Frequency Conversion.

Frequency up-converter.



Assume $\Delta k = 0$ (3 wave mixing equations) and that the pump remains approximately constant (very powerful beam) $\quad a_2(z) \simeq a_2(0)$.

$$\Rightarrow \quad \frac{da_1}{dz} = -j\frac{\gamma}{2}a_3 . \qquad \frac{da_3}{dz} = -j\frac{\gamma}{2}a_1 . \quad \Big\} \text{simple D.E.}$$

$\downarrow$ harmonic solutions.

$\gamma = 2g\,a_2(0)$ $\quad a_2(0)$ assumed real

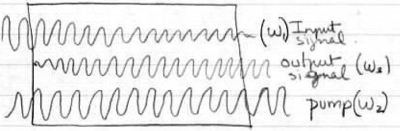
Solutions: $a_1(z) = a_1(0)\cos\frac{\gamma z}{2}$

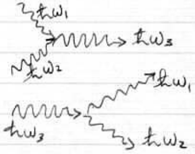
$$a_3(z) = -j a_1(0)\sin\frac{\gamma z}{2}.$$

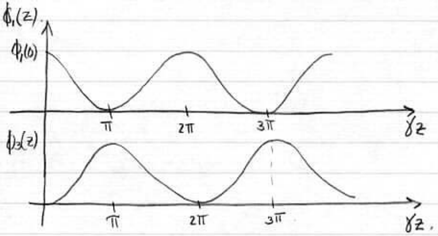
- Corresponding photon flux densities:

$$\phi_1(z) = \phi_1(0)\cos^2\left(\frac{\gamma z}{2}\right)$$

$$\phi_3(z) = \phi_1(0)\sin^2\left(\frac{\gamma z}{2}\right).$$







photons converted from $\omega_1 \to \omega_3$. (periodic).

$0 \le z \le \frac{\pi}{\gamma}$ $\quad \omega_1 \to \omega_3$ (ω_1 attenuated up-conversion.

$\pi/\gamma \le z \le \frac{2\pi}{\gamma}$ $\quad \omega_3 \to \omega_1$ (ω_3 attenuated).

Efficiency for up-conversion for a device of length L.

$$\frac{I_3(L)}{I_1(0)} = \frac{\omega_3}{\omega_1}\sin^2\frac{\gamma L}{2}.$$

- for $\gamma L \ll 1$

$$\frac{I_3(L)}{I_1(0)} \simeq \frac{\omega_3}{\omega_1}\left(\frac{\gamma L}{2}\right)^2 = \left(\frac{\omega_3}{\omega_1}\right) g^2 L^2 \phi_2(0) = 2\omega_3^2 L^2 d^2 \eta^3 I_2(0)$$

which finally means: $\frac{I_3(L)}{I_1(0)} = 2\eta_0^3 \omega_3^2 \frac{d^2}{n^3} \cdot \frac{L^2}{A} P_2$

A = cross-sectional area
$P_2 = I_2(0) A$ is the pump power

$\frac{d^2}{n^3}$ (material parameter).

Parametric Amplification and Oscillation.

Wave 1 – signal to be amplified.
Wave 3 – pump.
Wave 2 – idler – auxilary wave created by the interact process.

Pump @ $\hbar\omega_3 \rightarrow \hbar\omega_1 + \hbar\omega_2$. Assuming a large pump $a_3(z) = a_3(0)$

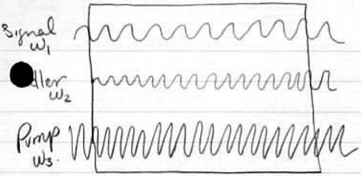
$$\Rightarrow \quad \frac{da_1}{dz} = -j\frac{\gamma}{2} a_2^* \qquad \frac{da_2}{dz} = -j\frac{\gamma}{2} a_1^*$$

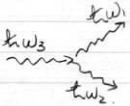
$\gamma = 2 g a_3(0)$. If $a_3(0)$ is real & γ is real

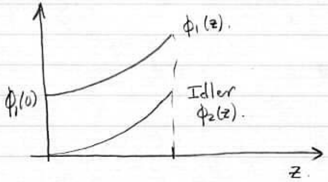
$$\Rightarrow \quad a_1(z) = a_1(0) \cosh\left(\frac{\gamma z}{2}\right)$$

$$a_2(z) = -j\, a_1(0) \sinh\left(\frac{\gamma z}{2}\right).$$

$$\phi_1(z) = \phi_1(0) \cosh^2\left(\frac{\gamma z}{2}\right) \qquad \phi_2(z) = \phi_1(0) \sinh^2\left(\frac{\gamma z}{2}\right).$$







Both $\phi_1(z)$ and $\phi_2(z)$ grow monotonically with z,
Saturates when sufficient energy is drawn from the pump so that $a_3(z) < a_3(0)$. $\Rightarrow$ assumption violated.

Total Gain for $\gamma L \gg 1$.
$$G = \frac{\left(e^{\gamma L/2} + e^{-\gamma L/2}\right)^2}{4} = \frac{\phi_1(L)}{\phi_1(0)}$$
$$= \cosh^2\left(\frac{\gamma L}{2}\right).$$

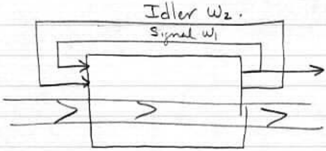
$\cancel{\gamma}$ $\quad \gamma L \gg 1 \Rightarrow \quad G \simeq e^{\gamma L}/4. \quad \Rightarrow$ gain exponential with γL.

$$\gamma = 2g\, a_3(0) = 2d\left(2\hbar\omega_1\omega_2\omega_3\eta^3\right)^{1/2} a_3(0).$$

$$\gamma = \left[8\eta_0^3\,\omega_1\omega_2\frac{d^2}{n^3}\frac{P_3}{A}\right]^{1/2}.$$ Parametric Amplifier Gain coefficient.

Parametric Oscillators.

Gain + Feedback.



$\rightarrow \quad \omega_3 = \omega_1 + \omega_2.$

To determine oscillation gain = loss. Losses not in the coupled wave equations.

We include the losses phenomenologically:

$$\frac{da_1}{dz} = -\frac{\alpha_1}{2} a_1 - j\frac{\gamma}{2} a_2^*$$

$$\frac{da_2}{dz} = -\frac{\alpha_2}{2} a_2 - j\frac{\gamma}{2} a_1^*.$$

α_1 & α_2 - power attenuation coefficients for signal and idler waves. (represent scattering and absorption losses in the medium and losses at mirrors of the resonator distributed along the length of the cavity).

for $\gamma = 0$ (no coupling).

$$\left.\begin{aligned} a_1(z) &= \exp\left(-\frac{\alpha_1 z}{2}\right) a_1(0) \\ \phi_1(z) &= \exp(-\alpha_1 z)\, \phi_1(0). \end{aligned}\right\}$$

photon flux decays at a rate α

Steady state solution:

$$0 = \alpha_1 a_1 + j\gamma a_2^*$$

$$0 = \alpha_2 a_2 + j\gamma a_1^*$$

$\Rightarrow \dfrac{a_1}{a_2^*} = \dfrac{-j\gamma}{\alpha_1}$ and $\dfrac{a_1}{a_2^*} = \dfrac{\alpha_2}{j\gamma}$ (conjugate eq. #2) 84

$$\Rightarrow \frac{-j\gamma}{\alpha_1} = \frac{\alpha_2}{j\gamma}$$

or $\gamma^2 = \alpha_1 \alpha_2$

If $\alpha_1 = \alpha_2 = \alpha$, condition for oscillation becomes.

$$\gamma = \alpha. \quad (\text{gain} = \text{loss}).$$

Since $\gamma = 2g a_3(0) \Rightarrow a_3(0) \geq \dfrac{\alpha}{2g}$ $\qquad \phi_3(0) \geq \dfrac{\alpha^2}{4g^2}$

$$g = (2\hbar \omega_1 \omega_2 \omega_3 \eta^3 d^2)^{1/2}$$

$$\Rightarrow \quad \phi_3(0) \geq \frac{\alpha^2}{8\hbar \omega_1 \omega_2 \omega_3 \eta^3 d^2}$$

$$\boxed{I_3(0) = \hbar\omega_3 \phi_3(0) \geq \frac{\alpha^2 n^3}{8\omega_1\omega_2 \eta_0^2 d^2}}$$ Parametric Oscillat Threshold pump intensity

Requires phase matching $n_1\omega_1 + n_2\omega_2 = n_2\omega_3$.
and frequency matching $\omega_1 + \omega_2 = \omega_3$

Dispersive medium $\Rightarrow$ $\underbrace{n_1(\omega_1)}_{\text{funct. of frequency}}$, $n_2(\omega_2)$, $n_3(\omega_3)$

$\Rightarrow$ Tuning accomplished by changing refractive indices.
(temperature tuning or angle tuning).

Coupled Wave Theory of Four-Wave Mixing

Four waves: $$E(t) = \sum_{q=1,2,3,4} \mathrm{Re}\left[E_q \exp(j\omega_q t)\right].$$

$$= \sum_{q=\pm1,\pm2,\pm3,\pm4} \frac{1}{2} E_q \exp(j\omega_q t).$$ with the same definit. $E_{-q} = E_q^*$ $\omega_{-q} = -\omega_q$.

Nonlinear Polarizat density

$$P_{NL} = 4\chi^{(3)} E^3$$

$$S = -\mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

$8^3 = 512$ terms!

$$S = \frac{1}{2}\mu_0 \chi^{(3)} \sum_{q,p,r=\pm1,\pm2,\pm3,\pm4} (\omega_q+\omega_p+\omega_r)^2 E_q E_p E_r \exp\left[j(\omega_q+\omega_p+\omega_r)t\right].$$

$\omega_1, \omega_2, \omega_3, \omega_4$ - four frequencies

Helmholtz equations

$$(\nabla^2 + k_q^2) E_q = -S_q \qquad q = 1,2,3,4.$$

S_q is the amplitude of the component of S at frequency ω_q.

Again, for the waves to be coupled if frequencies commensurate:

e.g. $$\omega_3 + \omega_4 = \omega_1 + \omega_2$$

3 Waves combine to generate a fourth wave.

Using $\omega_3 + \omega_4 = \omega_1 + \omega_2$ in S term.

$$S_1 = \mu_0 \omega_1^2 \chi^{(3)} \{ 6E_3E_4E_2^* + 3E_1[|E_1|^2 + 2|E_2|^2 + 2|E_3|^2 + 2|E_4|^2] \}$$

$$S_2 = \mu_0 \omega_2^2 \chi^{(3)} \{ 6E_3E_4E_1^* + 3E_2[|E_2|^2 + 2|E_1|^2 + 2|E_3|^2 + 2|E_4|^2] \}.$$

$$S_3 = \mu_0 \omega_3^2 \chi^{(3)} \{ 6E_1E_2E_4^* + 3E_3[|E_3|^2 + 2|E_2|^2 + 2|E_1|^2 + 2|E_4|^2] \}$$

$$S_4 = \mu_0 \omega_4^2 \chi^{(3)} \{ \underbrace{6E_1E_2E_3^*}_{\text{1st term}} + \underbrace{3E_4[|E_4|^2 + 2|E_1|^2 + 2|E_2|^2 + 2|E_4|^2]}_{\text{2nd term}} \}.$$

1st term: mixing of other 3 waves

2nd term: proportional to complex amplitude of the wave itself. (represents optical Kerr Effect).

We write this in a short-hand notation:

$$S_q = \overline{S_q} + (\omega_q/c_0)^2 \Delta\chi_q E_q \qquad q = 1, 2, 3, 4$$

Where

$$\overline{S_1} = 6\mu_0\omega_1^2 \chi^{(3)} E_3E_4E_2^*$$

$$\overline{S_2} = 6\mu_0\omega_2^2 \chi^{(3)} E_3E_4E_1^*$$

$$\overline{S_3} = 6\mu_0\omega_3^2 \chi^{(3)} E_1E_2E_4^*$$

$$\overline{S_4} = 6\mu_0\omega_4^2 \chi^{(3)} E_1E_2E_3^*$$

and $\Delta\chi_q = \dfrac{6\eta}{\epsilon_0} \chi^{(3)} (2I - I_q) \quad q = 1, 2, 3, 4.$

$I_q = \dfrac{|E_q|^2}{2\eta} \qquad I = I_1 + I_2 + I_3 + I_4$ = total intensity

η - impedance of the medium.

Helmholtz Eqn. $(\nabla^2 + \bar{k}_q^2) E_q = -\bar{S}_q \quad q = 1,2,3,4$

where $\bar{k}_q = \bar{n}_q \dfrac{\omega_q}{c_o}$

and $$\bar{n}_q = \left[n^2 + \frac{6\eta}{\epsilon_o} \chi^{(3)} (2I - I_q) \right]^{1/2} = n\left[1 + \frac{6\eta}{\epsilon_o n^2} \chi^{(3)} (2I - I_q)\right]^{1/2}$$

$$\simeq n\left[1 + \frac{3\eta}{\epsilon_o n^2} \chi^{(3)} (2I - I_q)\right]$$

from which:

$$\bar{n}_q = n + n_2 (2I - I_q) \qquad n_2 = \frac{3\eta_o}{\epsilon_o n^2} \chi^{(3)}$$

Helmholtz eqn modified:

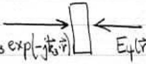
① source representy other 3 waves is present.
⇒ amplificat of wave or the emission of a new wave

② Refractive index for each wave is altered, becomes a function of the intensities of the four waves.

Four coupled differential eqns – solved under appropriate boundary conditions

Degenerate Four-Wave Mixing $\quad \omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega$

Waves 3 & 4 – called the pump waves: plane waves propagating in opposite directions.



$E_3(\vec{r}) = A_3 \exp(-j\vec{k}_3 \cdot \vec{r}) \qquad E_4(\vec{r}) = A_4 \exp(-j\vec{k}_4 \cdot \vec{r}) \qquad \vec{k}_4 = -\vec{k}_3.$

$A_3, A_4 >> A_2, A_1 \Rightarrow$ assume A_3 & A_4 are constant.

Total intensity is a constant

$$I \simeq \frac{\left[|A_3|^2 + |A_4|^2\right]}{2\eta}$$

$2I - I_1$ and $2I - I_2 \simeq 2I \Rightarrow \overline{n_q} = n\left[1 + \frac{3\eta}{\epsilon_0 n^2}\chi^{(3)} 2I\right] = \text{constant}$

$\Rightarrow$ Optical Kerr Effect results in a constant change in index.

With these assumptions, problem reduced to two coupled waves, 1 & 2.

$$(\nabla^2 + k^2)E_1 = -\xi E_2^*$$

$$(\nabla^2 + k^2)E_2 = -\xi E_1^*$$

where $\xi = 6\mu_0\omega^2\chi^{(3)}E_3E_4 = 6\mu_0\omega^2\chi^{(3)}A_3A_4$.

and $k = \frac{\overline{n}\omega}{c_0}$ where $\overline{n} \simeq n + 2n_2 I$ is a constant.

Four coupled differential equations have been reduced to two <u>linear</u> coupled equations $\Rightarrow$ Helmholtz. eqn with a source term.

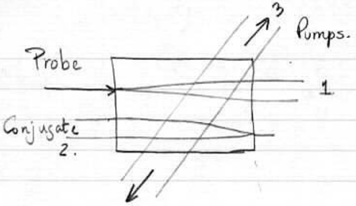
Source of wave 1 proportional to the conjugate of the complex conjugate of the complex amplitude of wave 2, and similarly for wave 2.

<u>Phase conjugation.</u>

If Waves 1 and 2 are also plane waves propagating in opposite directions along the z axis:

$$E_1 = A_1 \exp(-jkz) \quad ; \quad E_2 = A_2 \exp(+jkz)$$

$$\Rightarrow \quad k_1 + k_2 = k_3 + k_4.$$

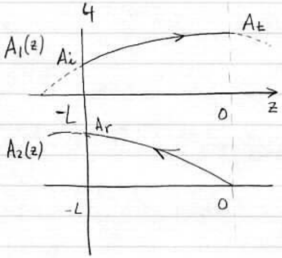


SVEA.

$$\Rightarrow \frac{dA_1}{dz} = -j\gamma A_2^*$$

$$\frac{dA_2}{dz} = j\gamma A_1^*$$

$$\gamma = \frac{\xi}{2k} = \frac{3\omega\eta_0\chi^{(3)}A_3A_4}{n}.$$



For simplicity: Assume A_3A_4 is real, $\Rightarrow \gamma$ is real

Solutions: two harmonic functions $A_1(z)$ and $A_2(z)$ with a 90° phase shift between them

Assume $A_1(-L) = A_i$ at entrance plane.
$A_2(0) = 0$

Under these boundary conditions

$$A_1(z) = \frac{A_i}{\cos\gamma L}\cos\gamma z$$

$$A_2(z) = j\frac{A_i^*}{\cos\gamma L}\sin\gamma z$$

Reflected wave at the entrance plane, $A_r = A_2(-L)$, is

$$A_r = -jA_i^*\tan\gamma L$$

Amplitude of the transmitted wave, $A_t = A_1(0)$, is

$$A_t = \frac{A_i}{\cos \gamma L}$$

Number of Applications

① Reflected wave is a conjugated version of the incident wave. The device acts as a phase conjugator.

② Intensity reflectance, $|A_r|^2/|A_i|^2 = \tan^2 \gamma L$ may be smaller or greater than 1. corresponding to attenuat or gain. Medium can be a reflection amplifier. (amplifying mirror)

③ Transmittance $\frac{|A_t|^2}{|A_i|^2} = \frac{1}{\cos^2 \gamma L}$ is greater than 1

$\Rightarrow$ transmission amplifier.

④ If $\gamma L = \frac{\pi}{2}$, or odd multiples thereof, the relfectance and transmittance are infinite $\Rightarrow$ instability $\Rightarrow$ oscillator.

Anisotropic Nonlinear Media.

$$P = (P_1, P_2, P_3)$$
$$E = (E_1, E_2, E_3)$$

$$P_i = \epsilon_0 \sum_j \chi_{ij} E_j + 2 \sum_{jk} d_{ijk} E_j E_k + 4 \sum_{jkl} \chi^{(3)}_{ijkl} E_j E_k E_l$$

$$i, j, k, l = 1, 2, 3.$$

χ_{ij}, d_{ijk}, $\chi^{(3)}_{ijkl}$ are elements of tensors χ, d, $\chi^{(3)}$ anisotropic media.

Symmetries: Coefficient d_{ijk} multiplies $E_j E_k \Rightarrow$ must be invariant to exchange of j & k.

- $\chi^{(3)}_{ijkl}$ invariant to any permutations of j, k, and l

$P_i = \epsilon_0 \sum_j \chi^e_{ij} E_j$ where χ^e_{ij} is the effective (field-dependent) tensor.

χ^e_{ij} – invariant to exchange of i and j

χ_{ij}, d_{ijk}, and $\chi^{(3)}_{ijkl}$: invariant to exchange of i and j

$\Rightarrow$ 3 tensors invariant to any permutations of their indices.

Use the same contraction notation as w/ Pockels and Kerr tensors $r_{ijk} \rightarrow r_{iK}$ and $s_{ijkl} \rightarrow s_{IK}$.

- Symmetries of d_{Ik} and $\chi^{(3)}_{IK}$ same as r_{Ik} & s_{IK}.

See Table (pg 780) attached.

<u>Three-wave Mixing in Anisotropic Second-Order Nonlinear Media.</u>

Suppose $E(t) = \sum_{q=\pm1,\pm2} E_q \exp(j\omega_q t)$. $\omega_3 = \omega_1 + \omega_2$.

$$P_i(\omega_3) = 2 \sum_{jk} d_{ijk} E_j(\omega_1) E_k(\omega_2) \qquad j, k = 1, 2, 3.$$

- $E_j(\omega_1)$, $E_k(\omega_2)$ and $P_i(\omega_3)$ are components of these vectors along the pricipal axes of the crystal.