Ray Tracing - path - only tells us the path

no amplitude, phase, spatial extent

→ Maxwell's equations ⇒ Approx soln wave equation.

→ Isotropic, Homogeneous, nonconducting medium.

curl eqns:

$$\nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t} = -\mu \frac{\partial \vec{h}}{\partial t} \quad (1)$$

$$\nabla \times \vec{h} = \frac{\partial \vec{d}}{\partial t} = \epsilon \frac{\partial \vec{e}}{\partial t} \quad (2)$$

$$\nabla \times \nabla \times \vec{e} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{h}) \qquad \nabla \times \nabla \times \vec{e} = \nabla(\nabla \cdot \vec{e}) - \nabla^2 \vec{e}$$

$$\nabla(\nabla \cdot \vec{e}) - \nabla^2 \vec{e} = -\mu\epsilon \frac{\partial^2 \vec{e}}{\partial t^2} \qquad \nabla \cdot \vec{e} = \rho = 0 \quad \text{free space.}$$

$$\nabla^2 \vec{e} - \frac{n^2}{c^2}\frac{\partial^2 \vec{e}}{\partial t^2} = 0. \qquad \mu\epsilon = \frac{n^2}{c^2}.$$

Assume n = constant.
harmonic time dependence : TEM

$$\vec{e}(x,y,z,t) = E(x,y,z)\, e^{j\omega t}\, \hat{a}_i \qquad \text{(for } x\text{-polarized).}$$

which means we must satisfy:

$$\nabla^2 E(x,y,z) + \frac{n^2\omega^2}{c^2} E(x,y,z) = 0 \qquad k = \frac{n\omega}{c}$$

$$\nabla^2 E(x,y,z) + k^2 E(x,y,z) = 0. \qquad \boxed{\text{Helmholtz equation}}$$

$$\nabla_T^2 E(x,y,z) + \frac{\partial^2}{\partial z} E(x,y,z) + k^2 E(x,y,z) = 0.$$

phase factors. $\sim e^{jkz} = \underbrace{e^{j\frac{2\pi n z}{\lambda}}}$ ↑ very rapidly varying function.

optical fields $\lambda \sim 1\mu m$

$$\rightarrow \qquad E(x,y,z) = E_0\, \psi(x,y,z)\, e^{jkz}$$
↑ rapidly varying.

$$\nabla_T^2 E(x,y,z) = E_0 (\nabla_T^2 \psi)\, e^{jkz}$$

$$\frac{\partial}{\partial z} E(x,y,z) = E_0\left(-jk\psi + \frac{\partial \psi}{\partial z}\right) e^{-jkz}$$

$$\frac{\partial^2}{\partial z^2} E(x,y,z) = E_0\left(-k^2\psi - 2jk\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2}\right) e^{-jkz}$$

Subst back into wave eqn, cancel E_0 & e^{-jkz} terms

$$\nabla_T^2 \psi - \cancel{k^2 \psi} - 2jk\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} + \cancel{k^2 \psi} = 0 .$$

$$\boxed{\nabla_T^2 \psi - 2jk \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0}$$ exact solution.

ψ slowly varying spatial envelope function.

$$\Rightarrow \frac{\partial \psi}{\partial z}, \frac{\partial^2 \psi}{\partial z^2} \text{ small}$$

$k = \frac{2\pi n}{\lambda} \sim 10^6 - 10^7$ for optical waves.

$2k \frac{\partial \psi}{\partial z} \gg \frac{\partial^2 \psi}{\partial z^2} \rightarrow \psi$ varies little over dimensions $\sim \lambda$.

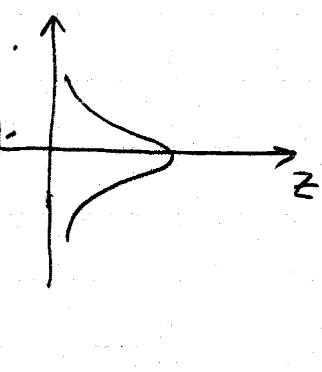
$$\Rightarrow \boxed{\nabla_T^2 \psi - 2jk\frac{\partial \psi}{\partial z} = 0.}$$

- Paraxial Wave Equation -
- central eqn for beam propagation.
- Central equation for Gaussian beams.
- (looks like time-dependent Schrödinger Equation).

Objective: generation, propagation, control } laser beams.

Gaussian intensity or field distribution (in transverse direction).
(cylindrical symmetry).

Look for solutions with cylindrical symmetry.
(TEM$_{00}$).



$$\nabla_T^2 \rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right).$$

wave eqn. $\rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - 2jk\frac{\partial \psi}{\partial z} = 0.$

How do we find solution to this DE.?

We guess! (as usual).

$$\psi = \exp\left\{-j\left(P(z) + \frac{k r^2}{2 q(z)}\right)\right\} \qquad \text{(pretty smart guess!)}$$

$$\frac{\partial \psi}{\partial z} = \left[-j P'(z) + \frac{j k r^2}{2 q^2(z)} q'(z)\right]\psi \qquad q'(z) = \frac{\partial q}{\partial z},\quad p'(z) = \frac{\partial p(z)}{\partial z}.$$

$$\frac{\partial}{\partial z}\left(\frac{1}{q(z)}\right) = \frac{q'(z)}{q^2(z)}.$$

$$\frac{\partial \psi}{\partial r} = \frac{-jkr}{q(z)}\psi.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = \left[-\frac{j2k}{q(z)} - \frac{k^2 r^2}{q^2(z)}\right]\psi.$$

Put into wave eqn:

$$\left(\left\{\frac{-k^2}{q^2(z)}[q'(z)-1]\right\} r^2 - 2k\left[p'(z) + \frac{j}{q(z)}\right] r^0\right)\psi = 0.$$

For this equation to hold for all r.

each coefficient of powers of r must equal zero

$$\Rightarrow \quad q'(z) = 1 \qquad p'(z) = \frac{-j}{q(z)}.$$

This leads to $q(z) = q_0 + z$ or $q(z) = z + jz_0$

q_0 ↑ complex constant. z_0 ↑ real constant

$\underbrace{\text{complex constant}}$ we know this from physical arguments.

$$P(z) = \int \frac{-j}{q(z)}\, dz$$

if $q(z)$ was real then

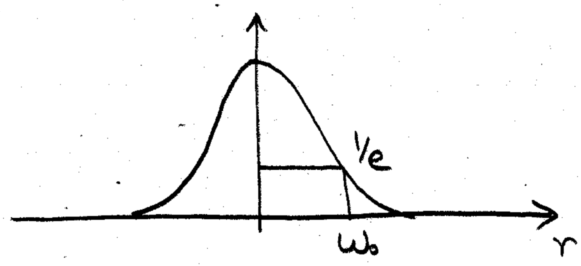
$\left|\exp\left(-\frac{jkr^2}{2q(z)}\right)\right| = 1$ amplitude constant.

$\Rightarrow$ phase is changing faster & faster with z not a beam.

$$\psi = \exp(-jP(z))\,\exp\left(-j\frac{k r^2}{2q(z)}\right).$$

Consider plane @ $z=0$. $\quad \psi(z=0) = e^{-jP(0)}\, e^{-\frac{jkr^2}{2jz_0}} = e^{-jP(0)}\, \underbrace{e^{-\left(\frac{r}{w_0}\right)^2}}_{\text{guassian}}$

w_0 — is called the spot size of the beam.



$$w_0^2 = \frac{2z_0}{k} = \frac{2z_0\lambda}{2\pi n} \longrightarrow z_0 = \frac{\pi w_0^2 n}{\lambda}. \qquad \text{confocal parameter}$$

$$\frac{1}{q(z)} = \frac{1}{z+jz_0} = \frac{z}{z^2+z_0^2} - j\frac{z_0}{z^2+z_0^2} = \frac{1}{R(z)} - j\frac{\lambda_0}{n\pi w^2(z)}$$

$$\psi(z) = \underbrace{e^{-\frac{kz_0 r^2}{2(z^2+z_0^2)}}}\, e^{-\frac{jkzr^2}{2(z^2+z_0^2)}}\, e^{-jP(z)}.$$

this gives amplitude profile: $e^{-\left(\frac{r}{w(z)}\right)^2} \qquad w^2(z) = \frac{2(z^2+z_0^2)}{kz_0}.$

We see: $\quad w^2(z) = \frac{2}{kz_0}(z^2+z_0^2) = \underbrace{\frac{2z_0}{k}}_{w_0^2}\left(1+\left(\frac{z}{z_0}\right)^2\right).$

$$= w_0^2\left(1+\left(\frac{z}{z_0}\right)^2\right). \Rightarrow \text{spot size changing with } z.$$

beam convergence or divergence

We define (second exponent) $R(z) = \frac{1}{z}(z^2+z_0^2) \qquad R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right).$

What is $P(z)$.

$P'(z) = \frac{-j}{q(z)} = \frac{-j}{z+jz_0} \qquad jP(z) = \int_0^z \frac{dz'}{z'+jz_0} = \ln(z'+jz_0)\Big|_0^{z}$

$$jP(z) = \ln\left[1 - j\left(\frac{z}{z_0}\right)\right]$$

rewrite: $$1 - j\left(\frac{z}{z_0}\right) = \left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2} e^{j\tan^{-1}\left(\frac{z}{z_0}\right)}.$$ Magnitude & phase.

$$jP(z) = \ln\left(1 + \left(\frac{z}{z_0}\right)^2\right)^{1/2} - j\tan^{-1}\left(\frac{z}{z_0}\right).$$

$$\Rightarrow e^{-jP(z)} = \frac{1}{\left(1 + \left(\frac{z^2}{z_0}\right)\right)^{1/2}} e^{-j\tan^{-1}\left(\frac{z}{z_0}\right)}$$

Finally, we write the general and fundamental solution to the wave equation ~~as~~ cylindrical symmetry.

$$E(x,y,z) = E_0\left\{\underbrace{\frac{w_0}{w(z)} \cdot e^{-\frac{r^2}{w^2(z)}}}_{\text{amplitude}} \underbrace{e^{-j\left(kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right)}}_{\text{longitudinal phase}} \underbrace{e^{-j\frac{kr^2}{2R(z)}}}_{\text{radial phase}}\right\}.$$

only assumptions ① Ψ doesn't vary much over λ
② cylindrical symmetry.

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]$$

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$z_0 = \frac{\pi n w_0^2}{\lambda_0}.$$

<u>What do these terms mean?</u>

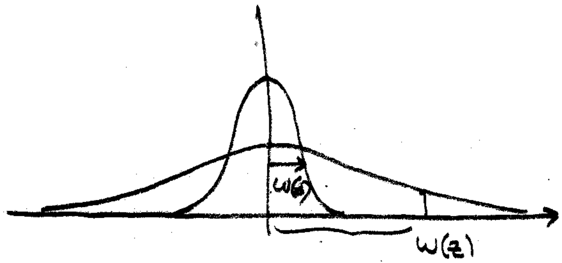
Amplitude: $\dfrac{w_0}{w_0^2\left(1 + \left(\frac{z}{z_0}\right)^2\right)} e^{-r^2/w^2(z)}$: radial distribution of field.
- variation with z.

$w(z) \rightarrow \frac{1}{e}$ point of field radial distribution

Notice that if $z = 0$

$\omega(0) = \omega_0$ beam waist. – minimum spot size for the given beam.

~~Hor~~ $\omega(z)$ describes the beam spread.



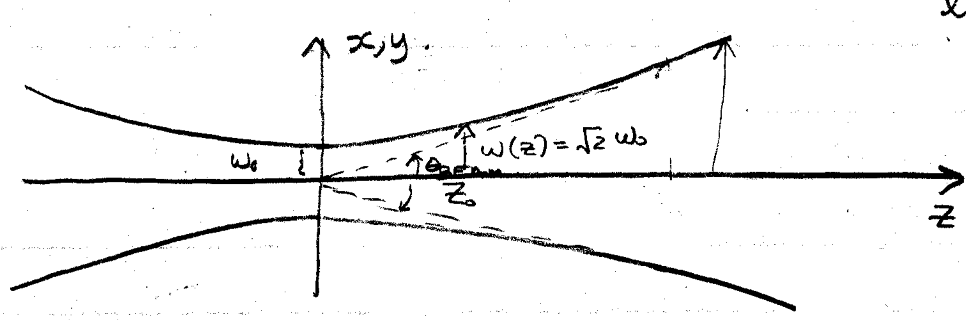
$$\omega(z) = \omega_0 \left[1 + \left(\frac{z}{z_0} \right)^2 \right]^{1/2}$$

for large z : $z \gg z_0$

$$\omega(z) \rightarrow \frac{\omega_0 z}{z_0}$$

$\Rightarrow \omega(z)$ increases linearly for large z.

If we plot the $\omega(z)$ as a funct. of z.



$$\tan \theta_{BEAM} = \frac{\omega(z)}{z} \; ; \text{ large } z \;\Rightarrow\; \theta_{BEAM} = \tan^{-1}\left(\frac{\omega_0}{z_0}\right).$$

typically $\omega_0 \ll z_0 \Rightarrow$, $\theta_{BEAM} = \dfrac{\omega_0}{z_0}$: half angle beam divergence.

$$z_0 = \frac{\pi n \omega_0^2}{\lambda} \Rightarrow \boxed{\theta_{BEAM} = \frac{\lambda}{\pi n \omega_0}}$$

start out with large beam it spreads out slower than small beam.

$\omega_0 \rightarrow \infty$ (planewave), $\theta_{BEAM} = 0$ plane waves don't diverge!

<u>Total Time Averaged Power crossing z = constant.</u>

$$P = \frac{1}{2} \iint \underbrace{\frac{E E^*}{\eta}}_{\text{Intensity}} dA = \frac{1}{2} \frac{E_0^2}{\eta} \frac{\omega_0^2}{\omega^2(z)} \int_0^{2\pi} \int_0^{\infty} r\,dr\,d\phi\, e^{-\left(\frac{2r^2}{\omega^2(z)}\right)}$$

$$= \frac{1}{2} \frac{E_0^2}{\eta} \left(\frac{\pi \omega_0^2}{2} \right) \text{ independent of } z.$$

Recall that we start with the wave equation:

$$\nabla^2 \vec{e} - \frac{n^2}{c^2}\frac{\partial^2 \vec{e}}{\partial t^2} = 0$$

$$\vec{e}(x,y,z,t) = E(x,y,z)\, e^{j\omega t} \longrightarrow \nabla^2 E(x,y,z) + \frac{n^2 \omega^2}{c^2} E(x,y,z) = 0$$

Assume $E(x,y,z) = E_0 \psi(x,y,z)\, e^{jkz}$.

$$\hookrightarrow \nabla_T^2 \psi - 2jk\frac{\partial \psi}{\partial z} = 0$$

cylindrical symmetry: $\psi = \exp\left\{-j\left(P(z) + \frac{kr^2}{2q(z)}\right)\right\}$.

$q(z) = z + jz_0$ $\qquad z_0 = \frac{\pi w_0^2 n}{\lambda}$ confocal parameter.

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda_0}{n\pi w^2(z)} \qquad w^2(z) = w_0^2\left(1 + \left(\frac{z}{z_0}\right)^2\right)$$

$$R(z) = z\left(1 + \left(\frac{z_0}{z}\right)^2\right)$$

$$P'(z) = \frac{-j}{q(z)}$$

$$\Rightarrow e^{-j(P(z))} = \frac{1}{\left(1 + \left(\frac{z}{z_0}\right)^2\right)^{1/2}} e^{-j\tan^{-1}\left(\frac{z}{z_0}\right)}$$

$$E(x,y,z) = E_0 \left\{ \underbrace{\frac{w_0}{w(z)} e^{\frac{-r^2}{w^2(z)}}}_{\text{amplitude}} \underbrace{e^{-j\left(kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right)}}_{\text{longitudinal phase}} \underbrace{e^{-j\frac{kr^2}{2R(z)}}}_{\text{radial phase}} \right\}$$

longitudinal phase

radial phase

Power is independent of $w(z)$. Total area under the curve is conserved. $\Rightarrow$ Amplitude decreases as $w(z)$ increases.

$$\theta_{BEAM} = \frac{\lambda}{\pi w_0 n}$$ optical beams $\lambda \sim 1\mu m$, $n=1$, $w_0 = 1mm$

$\Rightarrow \theta_{BEAM} = 0.3 mrad.$ very small divergence.

Take for example. $\lambda \sim 800nm$ $\lambda = 400nm$. same spot size $\Rightarrow$ divergence of shorter wavelength beam is ~1/2 as much.

Longitudinal phase. $\phi = kz - \tan^{-1}\left(\frac{z}{z_0}\right)$

For a plane wave phase velocity is $\phi_{pw} = kz \Rightarrow v_{pw} = c = \frac{2\pi f}{k}$. phase velocity.

$$v_{pw} = \frac{2\pi f z}{kz} = \frac{2\pi f z}{\phi_{pw}}$$

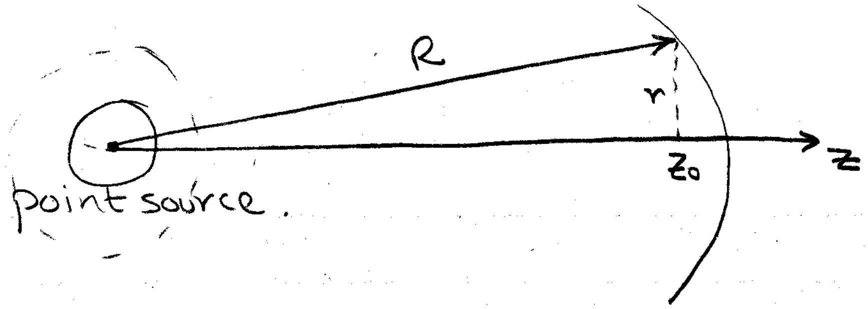
For Gaussian beam: $$v_G = \frac{2\pi \nu z}{\phi} = \frac{c}{1 - \underbrace{\frac{\lambda}{2\pi z}\tan^{-1}\left(\frac{z}{z_0}\right)}_{\text{small correctn.}}}$$

$z \to \infty$ $\frac{\lambda}{2\pi z}\tan^{-1}\left(\frac{z}{z_0}\right) \to 0 \Rightarrow v_G = c$ (at $z=\infty$ Gaussian $\to$ Plane wave).

$z \to 0$ $\frac{\lambda}{2\pi z}\tan^{-1}\left(\frac{z}{z_0}\right) \to \frac{\lambda}{2\pi z_0}$. $z_0 = \frac{\pi w_0^2 n}{\lambda} = 314cm$ for $n=1$, $w_0 = 1mm$, $\lambda = 1\mu m$

$$\frac{\lambda}{2\pi z_0} = 5\times10^{-8}$$ extremely small. correction.

Radial phase. $\frac{kr^2}{2R(z)}$. plane $z=$constant is not an equiphase plane as it is for a plane wave.



$$E \sim \frac{1}{R} e^{-ikR} \qquad R = \sqrt{r^2 + z_0^2}$$

look at points far from origin. $R \sim z_0 \gg r$.

$$R = z_0\left(1+\frac{r^2}{z_0^2}\right)^{1/2} \simeq z_0\left(1+\frac{r^2}{2z_0^2}\right) \doteq z_0 + \frac{r^2}{2z_0} = z_0 + \frac{r^2}{2R}$$

phase of spherical wave near z-axis is such

$$E \sim \frac{1}{R} e^{-ikz} \underbrace{e^{\frac{-kr^2}{2R}}}_{\text{spherical nature of phase front.}}$$

Same form as radial phase factor for Gaussian beam.

$\Rightarrow$ Gaussian $e^{-\frac{ikr^2}{2R(z)}}$ has spherical phase front with radius $R(z)$.

Gaussian $R = R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$.

$\Rightarrow$ apparent center of the phase front (i.e. R) is constantly changing. only for $z \gg z_0$ does the beam appear to have originated from the origin.

$z \gg z_0 \quad R(z) = z = R$

At $z = 0 \quad R(z) = z + \frac{z_0^2}{z^2} = \infty. \qquad R \to \infty$ plane wave.

Gaussian beam at $z=0$ looks like a plane wave.

Two equivalent descriptions of beam waist:

1. Minimum spot size, w_0
2. Planar phase front, $R \to \infty$.